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# Noisy One-Dimensional Maps Near a Crisis. I. Weak Gaussian White and Colored Noise

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We study one-dimensional single-humped maps near the boundary crisis at fully developed chaos in the presence of additive weak Gaussian white noise. By means of a new perturbation-like method the quasi-invariant density is calculated from the invariant density at the crisis in the absence of noise. In the precritical regime, where the deterministic map may show periodic windows, a necessary and sufficient condition for the validity of this method is derived. From the quasi-invariant density we determine the escape rate, which has the form of a scaling law and compares excellently with results from numerical simulations. We find that deterministic transient chaos is stabilized by weak noise whenever the maximum of the map is of order $z > 1$. Finally, we extend our method to more general maps near a boundary crisis and to multiplicative as well as colored weak Gaussian noise. Within this extended class of noises and for single-humped maps with any fixed order $z > 0$ of the maximum, in the scaling law for the escape rate both the critical exponents and the scaling function are universal.

**KEY WORDS:** Noisy map; crisis; escape rate; scaling and universality; invariant density; transient chaos; colored noise.

## 1. INTRODUCTION

Universality aspects as well as a more detailed quantitative understanding of particular systems close to the occurrence of qualitative changes in their behavior are of broad interest, for instance, in bifurcation theory, phase transitions, etc. In the field of nonlinear dissipative dynamics an important example of sudden qualitative changes is a crisis[1] where a chaotic attractor collides with a coexisting unstable fixed point or periodic orbit. While

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nonlinear deterministic models were successfully applied to a large variety of problems arising in very different scientific areas (see ref. 2 for a review), near a highly sensitive phenomenon like a crisis a realistic description often must include the effect of a small amount of noise.

The influence of weak noise on various low-dimensional dynamical systems near a crisis has been extensively studied for many years by analytical and numerical means,[3-14] whereas first experimental results have been reported only recently.[15-17] In this work we concentrate on noisy one-dimensional systems in discrete time. We mainly consider single-humped maps $f(x)$ with a maximum of arbitrary order $z>0$ close to the boundary crisis[1] at fully developed chaos.[18] Below the crisis, $f(x)$ maps the unit interval $[0, 1]$ into itself with a chaotic or, inside a periodic window, regular attractor on $[0, 1]$ and a second 'attractor' at minus infinity. At the crisis, the strange attractor collides with the unstable fixed point at $x=0$ and covers the whole unit interval. Beyond the crisis, one has a strange repeller instead of the attractor on $[0, 1]$, giving rise to transient chaos (see ref. 19 for a review). In the presence of weak but unbounded noise the unit interval becomes metastable both above and below the crisis. For large times the system approaches a quasistationary state described by a quasi-invariant density $W(x)$. A quantity of particular interest is the escape rate $k$ out of the unit interval in the quasistationary state. Whereas it is well known that the escape rate obeys a scaling law,[4] only approximate explicit results have been available so far even in the simplest case of additive Gaussian white noise.[3, 4, 8, 9, 11, 12] For a detailed discussion of the shortcomings or limitations of these results we refer to ref. 20 (see also refs. 11 and 21 and the comparison with numerical simulations in Tables I and II of the present paper).

We will present here a new method for the determination of this escape rate $k$ which becomes asymptotically exact for small noise strengths $\sigma$ and distances $\Delta$ from the crisis at $\Delta=0$ and compares very well with numerical simulations. As an important intermediate result the quasi-invariant density $W(x)$ also will be found. Moreover, generalizations to other kinds of maps $f(x)$ near a boundary crisis as well as to multiplicative and colored weak Gaussian noise will be given. Maps near interior crises and band-merging points as well as general (non-Gaussian) uncorrelated weak noise will be treated in a subsequent publication.[22] A preliminary version of our new method has already been used in a study of deterministic diffusion with noise[20] and of noise-induced escape from a point attractor with a fractal basin boundary.[21]

As an interesting consequence of our rate formula we will find that above the crisis the deterministic escape rate is always reduced by a sufficiently small amount of noise provided the maximum of the map $f(x)$ is of order $z > 1$. Thus deterministic transient chaos is stabilized by weak noise for $z > 1$ in agreement with a recent numerical investigation by Franaszek.[10] As a further remarkable conclusion we will find that for fixed $z$, in the scaling law for the escape rate both the critical exponents and the scaling function are universal for the entire class of single-humped maps and noises considered in this paper, including multiplicative and colored noise. While the universality of the exponents is not unexpected,[4, 11] the universality of the scaling function has not been observed previously. Similar universal scaling laws are well known from other routes to chaos[1, 2] described by one-dimensional maps with various kinds of noises. For a few representative investigations, dealing with the period-doubling and intermittency routes to chaos, see refs. 23 and 24 and refs. 13 and 15, respectively.

We proceed as follows: In the next section the single-humped maps $f(x)$ are specified in more detail and the basic equations governing the quasi-invariant density and the escape rate are derived. In Section 3 our new perturbation-like method is introduced which allows the determination of the quasi-invariant density close to the crisis and in the presence of weak noise from the invariant density at the crisis in the absence of noise. Below the crisis, the deterministic map shows periodic windows for $z > 1$, giving rise to considerable difficulties for any kind of perturbation theory, in particular in the limit of small noise strengths.[26] With these problems in mind, in Appendices A and B a sufficient and necessary condition for the validity of our new method is derived; see Eq. (41). In Section 4, the escape rate $k$ is determined from the quasi-invariant density, leading to the central scaling law (52). Sections 5 and 6 deal with the extension of the results for the quasi-invariant density and escape rate to more general maps and noises. The final Section 7 contains the summary and discussion.

## 2. THE MODEL

We consider the one-dimensional dynamics of a particle with coordinate $x$ in discrete time $n$ in the presence of additive Gaussian white noise

$$x_{n+1} = f(x_n) + \xi_n, \qquad P(\xi_n) = (2\pi\sigma^2)^{-1/2}\, e^{-\xi_n^2/2\sigma^2} \tag{1}$$

of small noise strength

$$0 \leqslant \sigma \ll 1 \tag{2}$$

As exemplified in Fig. 1, the function $f(x)$ in (1) is assumed to be a single-humped map of the real axis which is symmetric about its maximum of

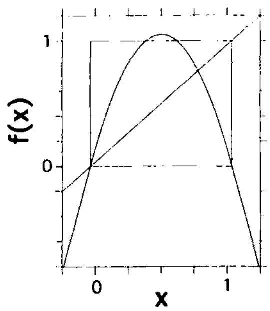


Fig. 1. The function $f(x)$ given by the logistic map (8) for $0 \leqslant x \leqslant 1$ and by the linear continuations (4) for $x \leqslant 0$ and (5) for $x \geqslant 1$ at the parameter value $\Delta = 0.05$.

order $z > 0$ at $x = 1/2$ and has an unstable fixed point at $x = 0$. Thus, close to the maximum the leading-order behavior is of the form

$$f(x) = 1 + \Delta - b\,|x - 1/2|^z \tag{3}$$

with $\Delta > -1, b > 0$, and $z > 0$. We further assume that $f(x)$ is continuously differentiable with $f'(x) \neq 0$ for all $x \neq 1/2$ and linear outside the unit interval,

$$f(x) = ux \qquad \text{for} \quad x < 0 \tag{4}$$

$$f(x) = u(1 - x) \qquad \text{for} \quad x > 1 \tag{5}$$

$$u := f'(0) > 1 \tag{6}$$

where the inequality in (6) follows from the fact that $x = 0$ is an unstable fixed point. More general maps will be considered in Section 5.

For $\Delta = 0$ the unit interval [0, 1] is mapped onto itself by $f(x)$ and there exists a unique invariant density $\rho(x)$ governing the deterministic dynamics (1) with $\sigma = 0$. We assume that $\rho(x)$ is positive, bounded, and continuous on the entire unit interval [0, 1] with the possible exceptions of arbitrarily small neighborhoods of the boundaries 0 and 1. To the best of our knowledge, conditions which guarantee these properties of $\rho(x)$ have not been studied in the literature and it is beyond the scope of the present paper to address this problem in detail. However, by closer inspection of the theoretical results and the examples in refs. 18 and 27 we are lead to the conjecture that a necessary and sufficient condition [in addition to those already imposed on $f(x)$] is either the absence of stable fixed points and periodic orbits of $f(x)$ at $\Delta = 0$ or (equivalently) that $\rho(x) \neq 0$ within an arbitrarily small neighborhood of $x = 1/2$. In particular, this will be guaranteed if a finite iterate of $f(x)$ is everywhere expanding or if $|f'(x)|^{-1/2}$ is a convex function, which means that the Schwarzian derivative of $f(x)$ is negative.

It follows that at $\Delta = 0$ we are dealing with a boundary crisis as defined by Grebogi *et al.*[1] and with fully developed chaos in the sense of Györgyi and Szépfalusy.[18] Thus, the parameter $\Delta$ in (3) represents the distance from the crisis at fully developed chaos and is always assumed to be small in the following,

$$-1 \ll \Delta \ll 1 \tag{7}$$

Note that unless otherwise stated, by 'small parameters' we mean both positive and negative $\Delta$ in (7), while 'small noise strengths' refers to non-negative $\sigma$ only; see Eq. (2).

Strictly speaking, we consider families of maps $f(x)$ smoothly parameterized by $\Delta$. While the exponent $z$ in (3) is required to be fixed, other properties of the map $f(x)$ such as $b$ in (3) and $u$ in (6) may still depend on the parameter $\Delta$. However, since we restrict ourselves to small $\Delta$ values, the variations of $b$ and $u$ are negligible and hence their dependence on $\Delta$ is dropped.

As mentioned in the introduction, on the unit interval [0, 1] the map $f(x)$ has a strange repeller giving rise to transient chaos above the crisis $\Delta > 0$. Below the crisis one either has permanent chaos on a strange attractor or, inside a periodic window, an asymptotically regular dynamics on a stable periodic orbit. However, one can show that the latter case of a periodic window is excluded for $z \leqslant 1$ and sufficiently small $\Delta$ by exploiting our assumptions regarding $f(x)$.

As a well-known example we mention the logistic map

$$f(x) = 4(1 + \Delta)\, x(1 - x) \tag{8}$$

with linear continuation (4)–(6) outside [0, 1]; see Fig. 1. Thus we have

$$z = 2, \qquad b = u = 4(1 + \Delta) \tag{9}$$

A further example is the tent map

$$f(x) = (1 + \Delta)(1 - 2\,|x - 1/2|) \tag{10}$$

with

$$z = 1, \qquad b = u = 2(1 + \Delta) \tag{11}$$

From the noisy dynamics (1) one finds for the transition probability $P(x|y)$ of a particle to get in one time step from $y$ to $x$ that

$$P(x \mid y) = \int_{-\infty}^{\infty} \delta(x - f(y) - \xi)\, P(\xi)\, d\xi = (2\pi\sigma^2)^{-1/2}\, e^{-[x - f(y)]^2/2\sigma^2} \qquad (12)$$

The probability densities $W_n(x)$ to be at $x$ after $n$ time steps follow from

$$W_{n+1}(x) = \int_{-\infty}^{\infty} P(x \mid y)\, W_n(y)\, dy \qquad (13)$$

As mentioned in the introduction, the system is expected to evolve toward a unique quasistationary state for large times $n$ independent of the initial density $W_0(x)$.[3,4] In particular, successive densities $W_n(x)$, $W_{n+1}(x)$ will not approach exact equality for large times $n$, but at least they become proportional to each other with a proportionality constant that must be $n$ independent according to (13):

$$W_{n+1}(x) = (1 - k)\, W_n(x) \qquad (14)$$

Evidently, the quantity $k$ plays the role of a decay rate.[2]

From (13) and (14) we can infer by integration that

$$k = \frac{\int_0^t [\, W_n(x) - \int_{-\infty}^{\infty} P(x \mid y)\, W_n(y)\, dy]\, dx}{\int_0^t W_n(x)\, dx} \qquad (15)$$

for arbitrary $t > 0$. For $t = 1$ the right-hand side of (15) represents the relative decrease of the population in $[0, 1]$ after one time step. Thus $k$ also has the meaning of an escape rate out of the unit interval in the quasistationary state. Similarly, choosing $t = \infty$, one can identify $k$ in (15) as the escape rate from $\mathbf{R}_+$ to $\mathbf{R}_-$. Henceforth, we will always use (15) with $t = \infty$ for the determination of the rate.

It is rather obvious and will be confirmed later by explicit results that for small noise strengths $\sigma$ and parameters $\Delta$ also the rate $k$ becomes small. Since the right-hand side of (15) with $t = \infty$ is given by the difference of two quantities of order one, it might seem that the densities $W_n(x)$ have to be known with extremely high accuracy in order to calculate the rate. However, three different arguments can be given which show that this is

[2] Strictly speaking, for a nonvanishing rate $k$, Eq. (14) cannot be rigorously true on the whole real axis, as follows, for instance, by comparison of (13) and (14) integrated over $x$. However, closer inspection shows that for any fixed $x_0 \in \mathbf{R}$, Eq. (14) becomes asymptotically exact for large times $n$ on the whole domain $x \geqslant x_0$. It is only in this sense that we actually will make use of (14).

actually not the case, basically thanks to the integration over $x$ in (15): First, one expects that the escape rate out of $\mathbf{R}_+$ approaches its asymptotic value $k$ even if the quasistationary state is only approximately reached, i.e., (14) is only fulfilled approximately. Second, since the decrease of the population of $\mathbf{R}_+$ after one time step is equal to the increase of population of $\mathbf{R}_-$, we can rewrite (15) as

$$k = \frac{\int_{-\infty}^{0} \left[ \int_{-\infty}^{\infty} P(x \mid y)\, W_n(y)\, dy - W_n(x) \right] dx}{\int_0^\infty W_n(x)\, dx} \tag{16}$$

For an initial distribution $W_0(x)$ which is concentrated in the unit interval, the population of $\mathbf{R}_-$ will stay small even for rather large times $n$ such that the quasistationary state is reached in good approximation. For such values of $n$, the rate in (16) becomes the difference of two small quantities and thus is supposed to be not too sensitive against small changes of $W_n(x)$. Third, by some straightforward manipulations, the rate (16) can be rewritten in the form[28, 29]

$$k = \frac{J_+ - J_-}{N_+} = \frac{\left[ \int_{-\infty}^{0} dx \int_0^\infty dy - \int_0^\infty dx \int_{-\infty}^{0} dy \right] P(x \mid y)\, W_n(y)}{\int_0^\infty W_n(x)\, dx} \tag{17}$$

where $N_+$ is the population of $\mathbf{R}_+$, while $J_+$ can be interpreted as a flux of particles escaping from $\mathbf{R}_+$ to $\mathbf{R}_-$ and $J_-$ as a flux of particles returning from $\mathbf{R}_-$ to $\mathbf{R}_+$. It is suggestive and can be confirmed by exploiting results later in this paper that the net flux $J_+ - J_-$ in (17) is not a small quantity in comparison with the partial fluxes $J_+$ and $J_-$ but rather is of the same order of magnitude. Thus the accuracy of the rate $k$ in (17) is actually comparable to the accuracy of $W_n(x)$ and consequently the same carries over to the equivalent rate formula (15) with $t = \infty$.

In summary, the problem to be solved in the next sections can be formulated as follows: In the quasistationary state characterized by (14), the densities $W_n(x)$ become proportional to a quasi-invariant density, $W_n(x) \propto W(x)$. With (13) and (14) we find that

$$(1-k)\, W(x) = \int_{-\infty}^{\infty} P(x \mid y)\, W(y)\, dy \tag{18}$$

and similarly to (15) this leads to the rate

$$k = \frac{\int_0^\infty \left[ W(x) - \int_{-\infty}^{\infty} P(x \mid y)\, W(y)\, dy \right] dx}{\int_0^\infty W(x)\, dx} \tag{19}$$

As seen in the preceding paragraph, in order to calculate the rate $k$ it is sufficient that (14) and hence (18) is fulfilled in good approximation. Since the rate $k$ in (18) is a small quantity, it thus suffices to find an approximate solution of the master equation

$$W(x) = \int_{-\infty}^{\infty} P(x \mid y)\, W(y)\, dy \tag{20}$$

instead of the exact quasi-invariant density in (18). In the next section such an approximate solution of (20) will be derived and in Section 4 the resulting rate (19) will be discussed. It turns out that our solution for $W(x)$ fulfills (20) in arbitrarily good approximation and the rate (19) becomes arbitrarily small for sufficiently small noise strengths $\sigma$ and parameters $\Delta$. Consequently, also the full equation (18) for the quasi-invariant density is strictly fulfilled and the rate (19) becomes exact for asymptotically small $\sigma$ and $\Delta$. Furthermore, it follows that our solution $W(x)$ asymptotically approaches the unique exact quasi-invariant density. We thus can omit any question concerning the uniqueness of $W(x)$ in the next section.

## 3. THE QUASI-INVARIANT DENSITY

In this section an approximate solution $W(x)$ of the master equation (20) for small noise strengths $\sigma$ and parameters $\Delta$ is constructed. This will be done in three steps: In the next subsection we determine the quasi-invariant density $W(x)$ in the domain $x \geqslant 1-\varepsilon$, where $\varepsilon$ is chosen much larger than $\sigma$ and $|\Delta|$ but sufficiently small such that the linearizations (4) and (5) can be used for all $x \leqslant \varepsilon$ and $x \geqslant 1-\varepsilon$, respectively. Then, in Sections 3.2 and 3.3 the quasi-invariant density $W(x)$ is determined in the domains $x \leqslant \varepsilon$ and $\varepsilon \leqslant x \leqslant 1-\varepsilon$, respectively.

### 3.1. Solution for $x \geqslant 1-\epsilon$

For $x \geqslant 1-\varepsilon$, the transition probability $P(x|y)$ in (12) is negligible for $y$ values outside a small neighborhood $[1/2-\delta, 1/2+\delta]$ of the maximum of $f(y)$ at $y = 1/2$; see also Fig. 1. Similar to $\varepsilon$, the quantity $\delta$ must be much larger than $\sigma^{1/z}$ and $|\Delta|^{1/z}$, but can still be chosen sufficiently small such that for $y \in [1/2-\delta, 1/2+\delta]$ the approximation (3) for $f(y)$ can be used in the transition probability (12). Next we make two assumptions regarding the quasi-invariant density $W(x)$ which can be shown to be consistent with the results for $W(x)$ found later in this section (although for the first of them this will not be worked out explicitly). First, we assume that $P(x|y)$ multiplied by $W(y)$ is still negligible for $x \geqslant 1-\varepsilon$ and $y$ outside $[1/2-\delta, 1/2+\delta]$. Second, we assume that $W(y)$ is sufficiently smooth on $[1/2-\delta, 1/2+\delta]$ such that it can be approximated by $W(1/2)$. [For instance, this assumption is obviously valid for all examples of $W(x)$

plotted in Fig. 3 except for (e) and (g). A rigorous justification will be given at the end of Section 3.3.] Then, for $x \geq 1-\varepsilon$ the master equation (20) takes the form

$$W(x) = W\left(\frac{1}{2}\right) \int_{-\delta}^{\delta} \exp\left\{-\frac{[x-1-\Delta+b\,|y|^z]^2}{2\sigma^2}\right\} \frac{dy}{(2\pi\sigma^2)^{1/2}} \tag{21}$$

Next we note that our assumptions so far on the quantities $\varepsilon$ and $\delta$ still allow a choice such that $f(1/2 \pm \delta) \simeq 1+\Delta-b|\delta|^z < 1-\varepsilon$. Then, the integrand in (21) takes its maximum in the interior of the integration domain $[-\delta, \delta]$ for any $x \geq 1-\varepsilon$. Thus, for sufficiently small $\sigma$ and $\Delta$ the integration can be extended over the whole real axis, yielding

$$W(1+x) = 2W\left(\frac{1}{2}\right) \int_0^{\infty} \exp\left\{-\frac{[x-\Delta+by^z]^2}{2\sigma^2}\right\} \frac{dy}{(2\pi\sigma^2)^{1/2}} \tag{22}$$

for $x \geq -\varepsilon$.

For the further discussion we rewrite (22) as

$$W(x) = \frac{2W(1/2)}{\sqrt{\pi}\, zb^{1/z}} (\sqrt{2}\sigma)^{(1-z)/z} H\left(\frac{1+\Delta-x}{\sqrt{2}\sigma}\right) \tag{23}$$

$$H(v) := \int_0^{\infty} y^{(1-z)/z} e^{-(y-v)^2}\, dy = \frac{\Gamma(1/z)}{2^{1/z}} e^{-v^2/2} D_{-1/z}(-\sqrt{2}v) \tag{24}$$

where $D_\nu(x)$ is the parabolic cylinder function (see, e.g., Eq. 3.462 in ref. 30). The function $H(v)$ is plotted in Fig. 2 for different values of $z$. From (24) one finds by closer inspection that $H(v)$ is strictly monotonically increasing on the whole real axis for $z \leq 1$. In particular, one has

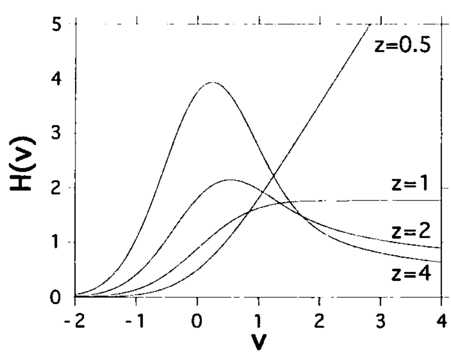


Fig. 2. The function $H(v)$ defined in (24) for different values of $z$.

$H(v) = (\sqrt{\pi}/2)\,\mathrm{erfc}(-v)$ for $z=1$. For $z>1$ the function $H(v)$ has a unique maximum at

$$v_{\max} = O(1/z) \qquad \text{with} \quad \max_v H(v) = H(v_{\max}) = O(z) \tag{25}$$

and tends to 0 for asymptotically large positive and negative $v$ like

$$H(v) = \sqrt{\pi}\, v^{(1-z)/z} \qquad \text{for} \quad v \gg 1 \tag{26}$$

$$H(v) = \Gamma\left(\frac{1}{z}\right) \frac{e^{-v^2}}{(2\,|v|)^{1/z}} \qquad \text{for} \quad v \ll -1 \tag{27}$$

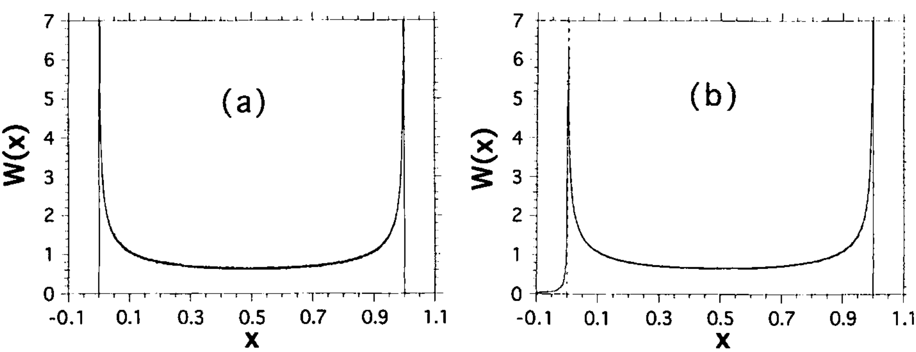


Fig. 3. The solid lines show invariant densities $W(x)$ from numerical simulations of the Langevin equation (1) for the logistic map (8) with linear continuation (4), (5) outside the unit interval. The roughness of the lines is due to the finite number of realizations. The noise strengths are $\sigma=0$ in (a), (c), (e), and (g) and $\sigma=0.0003$ otherwise. The parameter values are $\Delta=0$ in (a) and (b) corresponding to the boundary crisis of the map at fully developed chaos, $\Delta=0.002$ in (c) and (d) corresponding to transient chaos, $\Delta=0.002$ in (e) and (f) corresponding to permanent chaos, and $\Delta=0.00243$ in (g) and (h) corresponding to a periodic window of period 5. Thus the solid line represents the 'true' invariant density in (a), (e), and (g) and the quasi-invariant density otherwise. Note that in the plots with $\sigma=0$ all the peaks should actually be of infinite height. In contrast, for nonvanishing noise strength $\sigma$ the peaks are of finite height, but for better visibility those near $x=1$ are not shown in full height in the plots. The dashed lines are the theoretical invariant density for $\sigma=\Delta=0$, given by $\rho(x)=\{\pi[x(1-x)]^{1/2}\}^{-1}$ for $0\leqslant x\leqslant 1$, [see (43) and (45)] and $\rho(x)=0$ otherwise. For convenience, $W(x)$ is normalized on the unit interval and not according to $W(1/2)=\rho(1/2)$ as suggested from the discussion in Section 3. Thus, in (a) the (hardly visible) dashed line is a check of the numerical simulations. In the other plots the dashed lines confirm the theoretical prediction that $\rho(x)=W(x)$ for $\varepsilon\leqslant x\leqslant 1-\varepsilon$ [see (47)] provided the condition (41) is fulfilled. In particular, (41) is evidently violated in (e) and (g) and is reasonably but not extremely well fulfilled in (f) and (h), since we have $\sigma^{(z-1)/z}\simeq 8.71|\Delta|$ and $\sigma^{(z-1)/z}\simeq 7.1|\Delta|$, respectively. This explains the deviations from $\rho(x)=W(x)$ for $\varepsilon\leqslant x\leqslant 1-\varepsilon$ in (f) and (h), while the deviations in (b)–(d) are due to the finiteness of $\sigma$ and $\Delta$. A similar discussion applies to the theoretical predictions (28), (42), and possible deviations thereof. Finally, for $\Delta\neq 0$ also the predicted multipeak structure (33), (36) of $W(x)$ in the regime $x\leqslant\varepsilon$ is confirmed by (c), (d), (f), and (h).

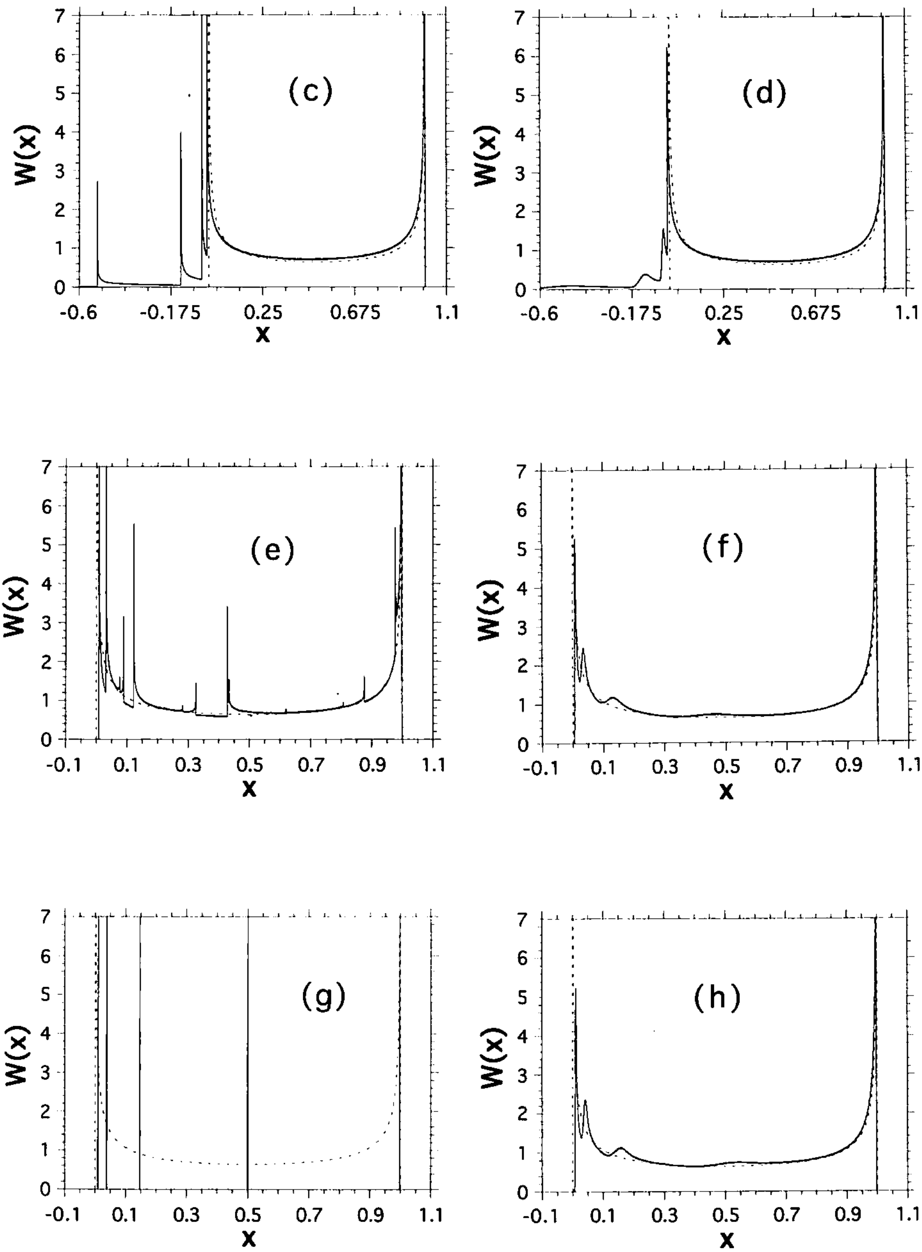


Fig. 3 (*continued*)

The same asymptotic expressions (26), (27) are actually valid also for $z \leqslant 1$. Consequently, the quasi-invariant density $W(x)$ in (23) has a maximum proportional to $1/\sigma^{(z-1)/z}$ at $x = 1 + \Delta - O(\sigma/z)$ for $z > 1$ and is strictly monotonically decreasing for $z \leqslant 1$. For any $z$, an exponential decay is approached for $x - (1 + \Delta) \gg \sigma$ and a behavior proportional to $(1 + \Delta - x)^{(1-z)/z}$ for $1 + \Delta - x \gg \sigma$. In the latter expression, $\Delta$ becomes negligible for $1 - x \gg |\Delta|$, implying

$$W(x) = B[1-x]^{(1-z)/z} \qquad \text{for} \quad \varepsilon \geqslant 1 - x \gg |\Delta| + \sigma \tag{28}$$

$$B := \frac{2W(1/2)}{zb^{1/z}} \tag{29}$$

See Fig. 3.

### 3.2. Solution for $x \leqslant \epsilon$

In the case that $x \leqslant \varepsilon$ the transition probability $P(x|y)$ in (12) is negligible for $\varepsilon \leqslant y \leqslant 1 - \varepsilon$; cf. Fig. 1. For $y \leqslant \varepsilon$ and $y \geqslant 1 - \varepsilon$ the map $f(y)$ in the transition probability (12) can be approximated by (4) and (5), respectively. As before, we make two assumptions regarding $W(x)$ which can be shown to be consistent with the results found later in this section. First, we assume that in the master equation (20) $P(x|y)$ multiplied by $W(y)$ is still negligible for $x \leqslant \varepsilon$ and $\varepsilon \leqslant y \leqslant 1 - \varepsilon$, yielding

$$\begin{aligned} W(x) = &\int_{-\infty}^{\varepsilon} W(y) \exp\left\{-\frac{[x - uy]^2}{2\sigma^2}\right\} \frac{dy}{(2\pi\sigma^2)^{1/2}} \\ &+ \int_{-\varepsilon}^{\infty} W(1+y) \exp\left\{-\frac{[x + uy]^2}{2\sigma^2}\right\} \frac{dy}{(2\pi\sigma^2)^{1/2}} \end{aligned} \tag{30}$$

The exponential term in the first integral has a very pronounced maximum at $y = x/u < \varepsilon$. Thus, apart from the factor $W(y)$, the integration domain could be extended over the whole real axis. Our second assumption is that this is in fact possible even if the factor $W(y)$ is included. After introducing (22) in the second integral in (30), one easily sees that again the integration can be extended over the whole real axis. We thus arrive at the following inhomogeneous integral equation for $W(x)$:

$$W(x) = \int_{-\infty}^{\infty} W(y) \exp\left\{-\frac{[x - uy]^2}{2\sigma^2}\right\} \frac{dy}{(2\pi\sigma^2)^{1/2}} + w_1(x) \tag{31}$$

$$w_1(x) := 2W\left(\frac{1}{2}\right) \int_{-\infty}^{\infty} \int_0^{\infty} \frac{dy\, dv}{2\pi\sigma^2} \exp\left\{-\frac{[x + uy]^2 + [y - \Delta + bv^z]^2}{2\sigma^2}\right\} \tag{32}$$

Note that this integral equation leads to a solution $W(x)$ on the whole real axis which, however, represents the quasi-invariant density only for $x \leqslant \varepsilon$. Similarly, the quantity $W(1/2)$ in (32) is the quasi-invariant density and not the solution of (31) at $x = 1/2$.

The integral equation (31) is an inhomogeneous Fredholm equation with a compact integral operator. Although the inhomogeneity $w_1(x)$ is unbounded for $z < 1$, this structure of (31) as well as the multipeak shape of $W(x)$ near $x = 0$ in Figs. 3c, 3d, 3f, and 3h suggest that we try an ansatz for $W(x)$ in the form of an infinite sum

$$W(x) = \lim_{m \to \infty} \sum_{i=1}^{m} w_i(x) \tag{33}$$

where the $w_i(x)$ are defined by the recursion relation

$$w_{i+1}(x) := \int_{-\infty}^{\infty} w_i(x) \exp\left\{-\frac{[x - uy]^2}{2\sigma^2}\right\} \frac{dy}{(2\pi\sigma^2)^{1/2}} \tag{34}$$

$$w_0(x) := 2W\left(\frac{1}{2}\right) \int_0^{\infty} \exp\left\{-\frac{[x + \Delta - by^z]^2}{2\sigma^2}\right\} \frac{dy}{(2\pi\sigma^2)^{1/2}} \tag{35}$$

Note that this recursion relation correctly reproduces $w_1(x)$ in (32). From (34) and (35) one finds by a straightforward calculation that

$$w_i(x) = \frac{2W(1/2)}{\sqrt{\pi} z b^{1/z} u^i} (\sigma U_i)^{(1-z)/z} H\left(\frac{x + u^i \Delta}{\sigma u^i U_i}\right) \tag{36}$$

$$U_i := \left(2 \frac{u^2 - u^{-2i}}{u^2 - 1}\right)^{1/2} \tag{37}$$

where $H(v)$ is defined in (24). For $x \leqslant \varepsilon$, Eq. (31) is equivalent to the master equation (20). With (33) and (34) this implies that for $x \leqslant \varepsilon$

$$W(x) - \int_{-\infty}^{\infty} P(x \mid y)\, W(y)\, dy = -\lim_{m \to \infty} w_m(x) \tag{38}$$

Exploiting (36) and the properties (25)–(27) of $H(v)$, it can be shown (see also the next paragraph) that for $x \leqslant \varepsilon$ the sum in (33) is bounded and the right-hand side of (38) vanishes. Thus the ansatz (33) together with (36) indeed is a solution of our problem for $x \leqslant \varepsilon$.

The properties of $w_i(x)$ in (36) are very similar to those of $W(x)$ discussed in the last paragraph of Section 3.1: One finds an exponential decay for large negative $x$ values, while

$$w_i(x) = B u^{-i/z} [x + u^i \Delta]^{(1-z)/z} \qquad \text{for} \quad x + u^i \Delta \gg \sigma u^i U_i \tag{39}$$

where $B$ is defined in (29). For $z \leqslant 1$ the functions $w_i(x)$ are strictly monotonically increasing, while for $z > 1$ there is a maximum at $x = u^i[-\Delta + O(\sigma U_i/z)]$. Close to this maximum one has

$$w_i(x) \simeq w_i(-u^i\Delta) = Cu^{-i}(\sigma U_i)^{(1-z)/z} \qquad \text{for} \quad |x + u^i\Delta| \leqslant O(\sigma u^i U_i) \tag{40}$$

where $C := 2W(1/2)\Gamma(1 + 1/2z)/(\sqrt{\pi}b^{1/z})$. The resulting behavior of the quasi-invariant density (33) for $x \leqslant \varepsilon$ is in agreement with the numerical examples of Fig. 3. Without going into further details we mention that for $\Delta \ll -\sigma$ our solution $W(x)$ for $x \leqslant \varepsilon$ fits into the generic structure of invariant densities near an unstable fixed point as found in refs. 29, 31, and 32.

In the Appendix A it is shown that for sufficiently small $\sigma$ and $\Delta$ and under the additional condition that

$$\Delta \gg -\sigma^{(z-1)/z} \qquad \text{or} \qquad z \leqslant 1 \tag{41}$$

the quasi-invariant density (33) takes the form

$$W(x) = \frac{B}{u^{1/z} - 1} x^{(1-z)/z} \tag{42}$$

for $x \in [\varepsilon/u, \varepsilon]$. The basic idea underlying the derivation of (41) in Appendix A is to find an integer $N \gg 1$ such that the contributions of $w_i(x)$ to the infinite sum (33) can be neglected for $i \geqslant N$. Moreover, Eq. (39) has to be valid for all $i < N$ and the quantity $u^i/\Delta$ on the right-hand side has to be negligible in comparison with $x$ for all $x \in [\varepsilon/u, \varepsilon]$. Then, one immediately finds the result (42). Condition (41) represents a sufficient condition that such an integer $N$ exists. In other words, the condition (41) says that for $z > 1$ and a fixed negative parameter $\Delta$, the noise strength $\sigma$ cannot be chosen arbitrarily small, whereas for $\Delta \geqslant 0$ or for $z \leqslant 1$ and arbitrary $\Delta$ the choice of $\sigma$ is not restricted. Again speaking differently, condition (41) means that for $z > 1$ and a fixed noise strength $\sigma$ the admitted parameter values $\Delta$ are limited from below.

While in Appendix A it is shown that (41) is a sufficient condition for (42), in Appendix B examples are given where (42) is definitely wrong in the regime $\Delta \ll -\sigma^{(z-1)/z}$. These examples consist of maps $f(x)$ with arbitrary $z > 1$ at parameter values $\Delta < 0$ which belong to periodic windows but can be arbitrarily close to zero. We conjecture that for these examples and parameters which fulfill neither $\Delta \ll -\sigma^{(z-1)/z}$ nor $\Delta \gg -\sigma^{(z-1)/z}$, Eq. (42) represents something in between a rather bad and a reasonably good approximation, but becomes really satisfactory only if

(41) is fulfilled. This is confirmed by the numerical example shown in Fig. 3(h). In other words, (41) is also a necessary condition for (42) to hold at least for a set of $\Delta$ values accumulating at $\Delta = 0$. Since the examples of Appendix B are not exotic constructions but typically occur for the class of maps $f(x)$ introduced at the beginning of Section 2, the condition (41) cannot be substantially relaxed without severely restricting the admitted maps $f(x)$. Even more, the numerical examples shown in Fig. 3e and 3f suggest that (41) is a necessary condition for (42) also for those $\Delta$ values not covered by the examples of Appendix B [including small, negative $\Delta$ for which $f(x)$ exhibits a strange attractor].

In summary, (41) is a sufficient and presumably also necessary condition for (42) to hold. Loosely speaking, it guarantees that at least in the domain $[\varepsilon, 1-\varepsilon]$ the singularities of the noise-free invariant density that occur both inside the periodic windows and for chaotic attractors when $z > 1$ and $\Delta < 0$ are sufficiently 'washed out' by the noise; see Figs. 3e–h. Apparently, the fact that for $z \leqslant 1$ the values of $\sigma$ and $\Delta$ are not restricted by (41) is somehow related to the absence of periodic windows, as mentioned below Eq. (7).

### 3.3. Solution for $\epsilon \leqslant x \leqslant 1-\epsilon$

In this section the quasi-invariant density $W(x)$ is determined in the domain $\varepsilon \leqslant x \leqslant 1-\varepsilon$ under the assumption that (42) holds. Thus, from now on we always restrict ourselves to values of $\sigma$ and $\Delta$ fulfilling (41). Let us first consider the special case that $\sigma = \Delta = 0$. Then it is impossible for a particle with dynamics (1) to escapes from [0,1]; see Fig. 1. Hence $W_{\Delta=\sigma=0}(0)$ is not a quasi-invariant, but a 'true' invariant density, i.e., a rigorous solution of the master equation (20), and actually coincides with $\rho(x)$ as introduced at the beginning of Section 2,

$$\rho(x) = W_{\sigma=\Delta=0}(x) \tag{43}$$

Since $\lim_{\sigma\to 0} P(x|y) = \delta(x - f(y))$ according to (12), the master equation for $\rho(x)$ (20) goes over into the Frobenius–Perron equation

$$\rho(x) = \sum_{f(y)=x} \frac{\rho(y)}{|f'(y)|} \tag{44}$$

We recall that $\rho(x)$ was required in Section 2 to be positive, bounded, and continuous for $\varepsilon \leqslant x \leqslant 1-\varepsilon$. Further, it can be shown that the results for $W(x)$ in the domains $x \geqslant 1-\varepsilon$ and $x \leqslant \varepsilon$ as found in Sections 3.1 and 3.2, respectively, stay valid for $\sigma = \Delta = 0$. Consequently, $\rho(x)$ is given by

(28) for $1-\varepsilon \leqslant x \leqslant 1$ and vanishes for $x>1$. Equation (39) implies that (42) is valid for $\rho(x)$ not only within $[\varepsilon/u, \varepsilon]$, but on the whole domain $0 \leqslant x \leqslant \varepsilon$, while for $x<0$ one has again $\rho(x)=0$. From (28) and (42) we recover the well-known result[18] that $\rho(x)$ has singularities of order $(1-z)/z$ at $x=0$ and $x=1$ and that a necessary condition for $\rho(x)$ to share the symmetry of $f(x)$, i.e., $\rho(1/2+x)=\rho(1/2-x)$, is $f'(0)=2^z$. Thus the latter situation, called double symmetry, is nongeneric. In passing we note that by comparing our results for $W(x)$ near $x=0$ and $x=1$ or for $x \to \infty$ and $x \to -\infty$, we can completely exclude double symmetry $W(1/2+x)=W(1/2-x)$ whenever $\Delta \neq 0$ or $\Delta \neq 0$, respectively; see also Fig. 3. For our examples of the logistic map (8) and the tent map (10) the invariant densities $\rho(x)$ on the unit interval are given by[27]

$$\rho(x)=\frac{1}{\pi} \frac{1}{[x(1-x)]^{1/2}} \tag{45}$$

see (Fig. 3a), and

$$\rho(x)=1 \tag{46}$$

respectively. Thus both examples represent the nongeneric case of double symmetry.

Next we will show that the quasi-invariant density $W(x)$ is given in arbitrarily good approximation by

$$W(x)=\rho(x) \qquad \text{for} \quad \varepsilon \leqslant x \leqslant 1-\varepsilon \tag{47}$$

provided $\sigma$ and $\Delta$ are sufficiently small and (41) is fulfilled; see Fig. 3. Thus $W(x)$ will be determined on the whole real axis insofar as $\rho(x)$ is considered to be known. In order to verify (47) we will show that this ansatz (47) provides a solution of the master equation (20) up to an arbitrarily small error. By closer inspection it can be shown that given (47) and (41), one recovers the same results for $W(x)$ in the domains $x \geqslant 1-\varepsilon$ and $x \leqslant \varepsilon$ as found in Sections, 3.1 and 3.2, respectively, except that $W(1/2)$ has to be replaced by $\rho(1/2)$ everywhere. Then, by comparing the properties of $\rho(x)$ discussed in the preceding paragraph with (28) and (42), one can infer that (47) actually holds at least for $\varepsilon/u \leqslant x \leqslant 1-\varepsilon/u$. From Eq. (47) and the behavior of $\rho(x)$ on $[\varepsilon, 1-\varepsilon]$ together with the results for $W(x)$ in the domains $x \geqslant 1-\varepsilon$ and $x \leqslant \varepsilon$ it follows that the master equation (20) can be evaluated in saddle point approximation for $\varepsilon \leqslant x \leqslant 1-\varepsilon$ and sufficiently small $\sigma$, giving

$$W(x)=\sum_{f(y)=x} \frac{W(y)}{|f'(y)|} \tag{48}$$

Since the $\Delta$ dependence of $f(y)$ is negligible for sufficiently small $\Delta$ and $\varepsilon/u \leqslant f(y) \leqslant 1-\varepsilon/u$ for all $\varepsilon \leqslant y \leqslant 1-\varepsilon$ (see Fig. 1), Eq. (44) shows that $W(x)$ in (47) solves (48) and thus the master equation (20) for $\varepsilon \leqslant x \leqslant 1-\varepsilon$.

## 4. THE ESCAPE RATE

By exploiting probability conservation $\int_{-\infty}^{\infty} P(x \mid y)\, dx = 1$ as implied by (12), we can rewrite the rate (19) in the form [see also Eq. (16)]

$$k = \frac{\int_{-\infty}^{0} \left[ \int_{-\infty}^{\infty} P(x \mid y)\, W(y)\, dy - W(x) \right] dx}{\int_{0}^{\infty} W(x)\, dx} \tag{49}$$

From the results of the preceding section it readily follows that the denominator in (49) asymptotically equals $\int_0^\infty \rho(x)\, dx = 1$ for small noise strengths $\sigma$ and parameters $\Delta$. By means of (38) this yields for the rate (49) that

$$k = \lim_{m \to \infty} \int_{-\infty}^{0} w_m(x)\, dx \tag{50}$$

and hence with (24), (36), and (47) that

$$k = \frac{2\rho(1/2)}{\sqrt{\pi z b^{1/z}}} \lim_{m \to \infty} \frac{(\sigma U_m)^{(1-z)/z}}{u^m}$$
$$\times \int_{-\infty}^{0} dx \int_{0}^{\infty} dy\, y^{(1-z)/z} \exp\left\{ -\left( y - \frac{x + u^m \Delta}{\sigma u^m U_m} \right)^2 \right\} \tag{51}$$

By performing the $x$ integration and then a partial integration over $y$ one finally arrives at

$$k = \rho\left(\frac{1}{2}\right) \left(\frac{U_\infty}{b} \sigma\right)^{1/z} F\left(\frac{1}{U_\infty} \frac{\Delta}{\sigma}\right) \tag{52}$$

$$F(x) := 2\pi^{-1/2} \int_0^\infty y^{1/z} e^{-(y-x)^2}\, dy \tag{53}$$

where $U_\infty = [2u^2/(u^2-1)]^{1/2}$ according to (37). *This is the central result of our paper*, becoming asymptotically exact for small noise strengths $\sigma$ and parameters $\Delta$ and under the necessary[3] and sufficient condition (41). While the invariant density at the maximum $\rho(1/2)$ is globally dependent on the map $f(x)$, the other properties $z$, $\Delta$, $b$ and $U_\infty$ of the map entering the rate are local ones. Let us also recall that $b$ and $U_\infty$ can be considered to be $\Delta$ independent as discussed in the paragraph below Eq. (7), while $\rho(1/2)$ and $z$ are $\Delta$ independent by definition. Thus, the rate (52), as a function of $\sigma$ and $\Delta$, has the form of a scaling law[4, 11] with both universal critical exponents and a universal scaling function $F(x)$ for any fixed $z$. As in the theory of critical phenomena,[33] *the specific properties of the map* $f(x)$ (*at* $\Delta = 0$) *enter the scaling law* (52) *only through the nonuniversal scaling amplitudes* $\rho(1/2)(U_\infty/b)^{1/z}$ and $1/U_\infty$ *within any universality class defined by a fixed* $z$.

The scaling function $F(x)$ in (53) is closely related to $H(v)$ in (24) and can equally well be expressed by a complementary error function (by means of a partial integration) or a parabolic cylinder function:[30]

$$F(x) = \int_0^\infty \operatorname{erfc}(y^z - x)\, dy$$
$$= [2^{(z-1)/z}/\pi]^{1/2}\, \Gamma(1+1/z)\, e^{-x^2/2} D_{-(1+z)/z}(-\sqrt{2}x) \qquad (54)$$

Similarly to the discussion of $H(v)$ below Eq. (24), one finds that $F(x)$ is monotonically increasing for all $z > 0$ and that

$$F(x) = \frac{\Gamma(1+1/z)}{\sqrt{\pi}\, 2^{1/z}} \frac{e^{-x^2}}{|x|^{1+1/z}} \qquad \text{for} \quad x \ll -1 \qquad (55)$$

$$F(0) = \pi^{-1/2}\Gamma\left(\frac{1}{2z} + \frac{1}{2}\right) \qquad (56)$$

$$F(x) = 2x^{1/z} \qquad \text{for} \quad x \gg 1 \qquad (57)$$

[3] For the derivation of the escape rate (52) not the full property (47), for which (41) is a (presumably) necessary condition, but basically only $\int_0^\infty W(x)dx = \int_0^\infty \rho(x)\, dx$ in (49) and

$$\int_{1/2-\delta}^{1/2+\delta} P(x \mid y)\, W(y)\, dy = \rho(1/2) \int_{1/2-\delta}^{1/2+\delta} P(x \mid y)\, dy$$

for $x \geq 1 - \varepsilon$ [see (21)] is needed [the other assumptions regarding $W(x)$ mentioned in the first paragraphs of Sections 3.1 and 3.2 are actually very mild and can be shown to hold even when (41) is violated]. However, we expect (41) to be still a necessary condition for the validity of these two properties of $W(x)$ and hence of the rate formula (52) at least for 'typical' small $\sigma$ and $\Delta$, even though for certain 'atypical' $\sigma$ and $\Delta$ these two properties and hence (52) might also hold 'by coincidence' even when (41) is violated.

Thus, for $\Delta<0$ and sufficiently small $\sigma$ [but still respecting (41)] the asymptotics (55) yields an Arrhenius law for the rate (52) as was to be expected. *The exponentially leading Arrhenius factor* $\exp\{-(\Delta/\sigma U_\infty)^2\}$ in this rate has been derived previously by Beale.[8] On the other hand, for $\Delta>0$ the asymptotics (57) leads to the obviously correct result $k=\rho(1/2)(\Delta/b)^{1/z}$ for the rate (52) in the deterministic limit $\sigma\to 0$.[19]

The theoretical rate (52) for the logistic map (8) and the tent map (10) is compared with results from numerical simulations in Tables I and II, respectively. For sufficiently small $\sigma$ and $\Delta$ the agreement is excellent. The same rate formula (52) except that the quantity $U_\infty$ is everywhere replaced by $\sqrt{2}$ has been derived in ref. 3 [see Eq. (4.4) therein]. In other words, this result from ref. 3 is independent of $u$ and since $U_\infty=[2u^2/(u^2-1)]^{1/2}$, we see that it agrees with ours, (52), only in the limit $u\to\infty$, i.e., for maps $f(x)$ with a superunstable fixed point or a discontinuity at $x=0$ and symmetrically at $x=1$. For finite $u$, the numerical results of Tables I and II clearly favor our formula (52). Essentially the same restriction to $u=\infty$ applies to the results derived in refs. 4, 9, 11, and 34 as discussed in more detail in refs. 20–22.

**Table I. Comparison of the Theoretical Escape Rate $k_{th}$ in (52) with Results $k_{num}$ from Numerical Simulations of the Langevin Equation (1) for the Logistic Map (8) with Linear Continuation (4), (5) Outside the Unit Interval**[a]

| $\Delta/\sigma$ | $\sigma=10^{-2}$ | $\sigma=10^{-3}$ | $\sigma=10^{-4}$ | $\sigma=10^{-5}$ | $\sigma=10^{-6}$ |
|---|---|---|---|---|---|
| 2 | $1.176\times10^{-1}$ | $3.096\times10^{-2}$ | $9.114\times10^{-3}$ | $2.775\times10^{-3}$ | $8.615\times10^{-4}$ |
| | 27 (27) | 12 (12) | 5.5 (5.3) | 1.8 (1.6) | 0.0 (−0.3) |
| 0 | $3.151\times10^{-2}$ | $8.959\times10^{-3}$ | $2.702\times10^{-3}$ | $8.486\times10^{-4}$ | $2.682\times10^{-4}$ |
| | 16 (17) | 6.1 (7.6) | 1.6 (3.1) | 0.9 (2.5) | 0.8 (2.4) |
| −2 | $1.212\times10^{-3}$ | $3.239\times10^{-4}$ | $9.605\times10^{-5}$ | $3.037\times10^{-5}$ | $9.424\times10^{-6}$ |
| | 20 (34) | 7.4 (22) | 1.4 (17) | 1.4 (17) | −0.5 (16) |

[a] The first of each pair is the numerical rate $k_{num}$ for different noise strengths $\sigma$ and parameters $\Delta$ fulfilling the condition (41). The numbers below $k_{num}$ are the relative difference in percent, $100(k_{num}-k_{th})/k_{num}$, between theoretical and numerical rates. The theoretical rate (52) is completely fixed by (9), (45), and (53). The statistical uncertainty of the numerical rate due to the finite number of realizations is about 1%. Within this accuracy, the agreement between numerics and theory is perfect for sufficiently small $\Delta$ and $\sigma$. The numbers in parentheses represent the same relative difference in percent but with $U_\infty$ in the rate formula (52) replaced by $\sqrt{2}$ according to the result for $k_{th}$ derived in ref. 3 [see Eq. (4.4) therein]. The agreement with the numerical simulations is considerably worse, in particular for large negative $\Delta$, which can be understood by the fact that even the exponentially leading Arrhenius factor in the theoretical rate[3] is not correct; see the discussion below Eq. (57).

Table II. Same as Table I, but for the Tent Map (10)

| $\Delta/\sigma$ | $\sigma=10^{-1}$ | $\sigma=3.16\times10^{-2}$ | $\sigma=10^{-2}$ | $\sigma=3.16\times10^{-3}$ | $\sigma=10^{-3}$ |
|---|---|---|---|---|---|
| 2 | $1.959\times10^{-1}$ | $6.431\times10^{-2}$ | $2.007\times10^{-2}$ | $6.397\times10^{-3}$ | $2.015\times10^{-3}$ |
| | 14 (15) | 6.7 (7.1) | 1.4 (1.9) | 0.8 (1.3) | 0.0 (0.5) |
| 0 | $5.441\times10^{-2}$ | $1.547\times10^{-2}$ | $4.793\times10^{-3}$ | $1.445\times10^{-3}$ | $4.676\times10^{-4}$ |
| | 15 (27) | 5.8 (18) | 3.9 (17) | −0.8 (13) | 1.5 (15) |
| −2 | $6.869\times10^{-3}$ | $9.404\times10^{-4}$ | $2.279\times10^{-4}$ | $6.652\times10^{-5}$ | $2.001\times10^{-5}$ |
| | 41 (85) | 21 (70) | 9.3 (62) | 5.5 (59) | 1.9 (57) |

In the remainder of this section we consider the behavior of the rate for a fixed parameter $\Delta>0$ as a function of the noise strength $\sigma$. To this end we rewrite the scaling law (52) as

$$k=\rho\left(\frac{1}{2}\right)\left(\frac{|\Delta|}{b}\right)^{1/z} G\left(U_\infty\frac{\sigma}{\Delta}\right) \tag{58}$$

$$G(x):=2\pi^{-1/2}\int_{-1/x}^{\infty} e^{-y^2}\,|1+xy|^{1/z}\,dy \tag{59}$$

Closer inspection[22] shows that for $z\leqslant 1$ the scaling function $G(x)$ in (59) is strictly monotonically increasing with $x$, while for $z>1$ it has a local minimum at $x_{\min}(z)>0$; see also Fig. 4. The minimum $x_{\min}(z)$ is strictly monotonically increasing with $z$ and obeys $x_{\min}(z\downarrow 1)=0$ and $x_{\min}(z)\propto z$ for large $z$. Further, one finds that $G(x\downarrow 0)=2$ independent of $z$ and that $G(x_{\min}(z))$ is strictly monotonically decreasing from 2 for $z=1$ toward 1 for $z\to\infty$. Thus, for any $z>1$ and $\Delta>0$, sufficiently small noise strengths $\sigma>0$ will lead to smaller rates (58) than in the absence of noise $\sigma=0$, i.e.,

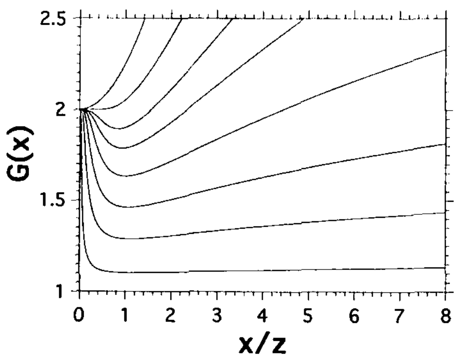


Fig. 4. The scaling function $G(x)$ defined in (59) versus $x/z$ for $z=0.5$, 1, 1.5, 2, 3, 6, 15, and 40 (from above).

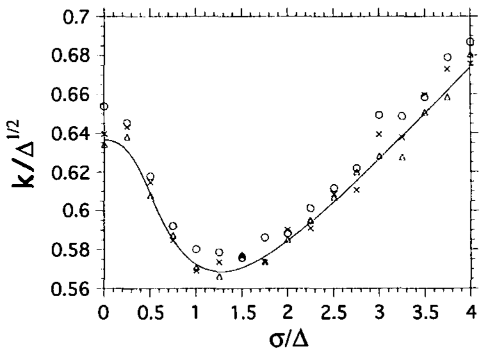


Fig. 5. The escape rate $k$ as a function of the noise strength $\sigma$ for the logistic map (8) with linear continuation (4), (5) outside the unit interval. The solid line is the theoretical prediction according to (9), (45), and (58). The symbols are results from numerical simulations of the Langevin equation (1) at parameter values $\Delta = 10^{-5}$ (circles), $\Delta = 10^{-6}$ (crosses), and $\Delta = 10^{-7}$ (triangles). The statistical error is about 1%. As seen in Table I, the agreement between theory and simulations is excellent, and, in particular, the predicted noise-induced stabilization of deterministic transient chaos is confirmed by the numerical results. Only for $\Delta = 10^{-5}$ are the numerical results systematically slightly above the theoretical line, which clearly is a finite-$\Delta$ and $\sigma$ effect.

*the noise induces a stabilization of deterministic transient chaos*; see Fig. 5. The strongest relative reduction $G(x_{\min}(z))/2$ of the rate only depends on $z$ but not on $\Delta$ and takes the extremal value 1/2 for $z \to \infty$, while for $z \leqslant 1$ no reduction occurs. The stabilization of deterministic transient chaos by noise has been observed in a numerical study by Franaszek.[10] A simple intuitive explanation of this effect is possible only in the limit $u \to \infty$, as discussed in more detail in refs. 20 and 22.

## 5. MORE GENERAL MAPS AND MULTIPLICATIVE NOISE

We consider a generalization of the Langevin equation (1) with *multiplicative noise*

$$x_{n+1} = f(x_n) + g(x_n)\, \xi_n \tag{60}$$

The map $f(x)$ is of the same kind as those introduced in Section 2 except that it must no longer be symmetric about $x = 1/2$. In particular, we still assume that $f(1) = 0$ without loss of generality. Accordingly, the quantity 1/2 in (3) has to be replaced by a general maximum $x^* \in (0, 1)$ of $f(x)$ and $u$ in (4) and (5) by $f'(0) > 1$ and $-f'(1) > 0$, respectively. The noise-coupling function $g(x)$ in (60) is required to be nonvanishing at least at one of the points $x=0$, $x=x^*$, or $x=1$ and to be continuous and bounded on the whole real axis. For convenience, we also assume for the moment that $g(x)=g(0)$ for $x\leqslant 0$ and $g(x)=g(1)$ for $x\geqslant 1$. The resulting generalizations for the quasi-invariant density $W(x)$ in Section 3 are straightforward and not elaborated in further detail here. As in Section 4, one then arrives at the escape rate

$$k=\rho(x^*)\left(\frac{\hat{U}}{b}\sigma\right)^{1/z} F\left(\frac{1}{\hat{U}}\frac{\Delta}{\sigma}\right) \tag{61}$$

for small $\sigma$ and $\Delta$ obeying (41), where the scaling function $F(x)$ is given in (53) and

$$\hat{U}:=\left\{2\left[g(x^*)^2+\frac{g(1)^2}{f'(1)^2}+\frac{g(0)^2}{f'(1)^2\,(f'(0)^2-1)}\right]\right\}^{1/2} \tag{62}$$

Clearly, for maps $f(x)$ which are symmetric about $x=1/2$ and additive noise $g(x)\equiv 1$ the rate formula (52) is recovered. Again, for any fixed order of the maximum $z$, both the critical exponents and the scaling function $F(x)$ in (61) are universal and the specific properties of the map $f(x)$ and the noise-coupling function $g(x)$ only enter into the scaling amplitudes $\rho(x^*)(\hat{U}/b)^{1/z}$ and $1/\hat{U}$. Thus, the further discussion of the rate (61) can be immediately taken over from Section 4.

The result (61) for the escape rate stays valid if the assumptions that $f(x)$ is linear and $g(x)$ is constant outside the unit interval are given up. Essentially, this follows from the fact that within a small neighborhood of the unit interval, say $[-\varepsilon, 1+\varepsilon]$, the dynamics (60) stays practically unchanged, while outside this neighborhood the probability that a realization returns into the unit interval is negligible anyway. The quasi-invariant density $W(x)$ stays unchanged as well, except that the functions $w_i(x)$ in the infinite sum (33) can no longer be expressed in a simple closed form as in (36) in the domain $x\leqslant -\varepsilon$.

As a further generalization we consider *maps* $f(x)$ *in* (60) *with* $M\geqslant 1$ *local extrema* $x_i^*$, $i=1, 2, ..., M$, that are continuously differentiable and obey $f'(x)\neq 0$ for all $x\neq x_i^*$. Without loss of generality we can assume that $0<x_1^*<x_2^*<\cdots<x_M^*<1$, that $x_i^*$ with odd and even $i$ values are local maxima and minima of $f(x)$, respectively, and that $f(0)=0$ and $f(1)=0$ ($M$ odd) or $f(1)=1$($M$ even). Analogous to (3), close to the extrema $x_i^*$ the map $f(x)$ is assumed to behave as

$$f(x)=1+c_i\Delta-b_i\,|x-x_i^*|^{z_i} \qquad \text{for odd } i \tag{63}$$

$$f(x)=-c_i\Delta+b_i\,|x-x_i^*|^{z_i} \qquad \text{for even } i \tag{64}$$

where $b_i > 0$, $z_i > 0$, and $c_i$ are arbitrary. Thus, for $\Delta = 0$ the unit interval is mapped onto itself and, similarly as in Section 2, we expect that a necessary and sufficient additional condition to guarantee a nicely behaving invariant density $\rho(x)$ is the absence of stable fixed points and periodic orbits or (equivalently) $\rho(x) \neq 0$ close to at least one extremum $x_i^*$.

For $M$ odd it follows that $x = 0$ is an unstable fixed point, $f'(0) > 1$, and $f(1) = 0, f'(1) < 0$. The determination of the quasi-invariant density $W(x)$ follows the same line of reasoning as in Section 3 and for the escape rate one finds that

$$k = \sum_{i=1}^{M} \rho(x_i^*) \left(\frac{\hat{U}_i}{b_i}\sigma\right)^{1/z_i} F_i\left(\frac{c_i}{\hat{U}_i}\frac{\Delta}{\sigma}\right) \tag{65}$$

where $F_i(x)$ is the scaling function in (53) with $z = z_i$ and

$$\hat{U}_i := \left\{2\left[g(x_i^*)^2 + \frac{g(1)^2}{f'(1)^2} + \frac{g(0)^2}{f'(1)^2\,(f'(0)^2 - 1)}\right]\right\}^{1/2} \quad \text{for odd } i \tag{66}$$

$$\hat{U}_i := \left\{2\left[g(x_i^*)^2 + \frac{g(0)^2}{f'(0)^2 - 1}\right]\right\}^{1/2} \quad \text{for even } i \tag{67}$$

For $M$ even, both $x = 0$ and $x = 1$ are unstable fixed points of $f(x)$ and there exist two partial escape rates $k_+$ and $k_-$ out of the unit interval $[0,1]$, describing particles which escape to plus and minus infinity, respectively. Clearly, the total escape rate is $k = k_+ + k_-$. The partial escape rate $k_-$ turns out to be equal to the right-hand side of (65) except that the sum runs over even $i$ only and consequently the $\hat{U}_i$ are given by (67). Similarly, $k_+$ is equal to the right-hand side of (65) with odd $i$ only and the $\hat{U}_i$ are given by (67) but with $f'(1)$ and $g(1)$ instead of $f'(0)$ and $g(0)$, respectively.

Clearly, there are many other possible generalizations, such as discontinuous functions $f(x)$ and $g(x)$, which, however, are not considered in further detail here.

## 6. COLORED NOISE

In generalization of (1) and (2) we consider the two-dimensional Langevin equation

$$x_{n+1} = f(x_n) + y_n + \xi_n^{(1)} \tag{68}$$

$$y_{n+1} = Ay_n + \xi_n^{(2)} \tag{69}$$

where the map $f(x)$ is specified in Section 2 and $\xi_n^{(i)}$, $i=1,2$, represents uncorrelated weak Gaussian noise

$$P(\xi_n^{(i)}) = (2\pi\sigma_i^2)^{-1/2} \exp(-\xi_n^{(i)^2}/2\sigma_i^2)$$
$$\langle \xi_n^{(i)}\xi_m^{(j)} \rangle = \sigma_i^2 \delta_{ij}\delta_{nm}, \qquad 0 \leqslant \sigma_i \ll 1 \tag{70}$$

Restricting ourselves to $A$ values in (69) with

$$-1 < A < 1 \tag{71}$$

it follows from (69) and (70) that in the stationary state $y_n$ are Gaussian distributed random numbers of vanishing mean and exponentially decreasing correlation

$$\tilde{p}(y_n) = \left(\frac{1-A^2}{2\pi\sigma_2^2}\right)^{1/2} \exp\left(-\frac{1-A^2}{2\sigma_2^2} y_n^2\right) \tag{72}$$

$$\langle y_n y_m \rangle = A^{|n-m|} \frac{\sigma_2^2}{1-A^2} \tag{73}$$

In other words, $y_n$ represents Ornstein–Uhlenbeck noise of correlation time $[\ln |A^{-1}|]^{-1}$. In particular, the white-noise limit is given by $A \to 0$. Note that Ornstein–Uhlenbeck noise in discrete time (69) shows a richer behavior than in continuous time (see, e.g., contributions in ref. 35), since it includes anticorrelated noise in (73) for negative $A$; see also ref. 36. The influence of Ornstein–Uhlenbeck noise on an experimental system near a crisis has recently been studied in ref. 16.

The one-dimensional dynamics (68), in which we are actually interested, is simultaneously disturbed by white and colored noise and, in particular, is non-Markovian for nonvanishing $\sigma_2$ and $A$. However, for practical purposes it is usually much more convenient to consider the two-dimensional Markov process (68), (69) and to eliminate the auxiliary variable y only at the very end by integration.[36] Thus, in generalization of (20), we will start with the determination of the two-dimensional quasi-invariant density $W(x, y) = W(x_1, x_2)$ by constructing an approximate solution of the master equation

$$W(x_1, x_2) = \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dy_2\, P(x_1, x_2 \mid y_1, y_2)\, W(y_1, y_2) \tag{74}$$

Similarly as in (12), the two-dimensional transition probability in (74) is found to read

$$P(x_1, x_2 \mid y_1, y_2) = \frac{1}{2\pi\sigma_1\sigma_2} \exp\left\{-\frac{[x_1 - f(y_1) - y_2]^2}{2\sigma_1^2} - \frac{[x_2 - Ay_2]^2}{2\sigma_2^2}\right\} \tag{75}$$

In a second step, we will use the approximate quasi-invariant density $W(x_1, x_2)$ to calculate the escape rate $k$, which in generalization of (19), now is given by

$$k = \frac{\left( \begin{array}{l} \int_0^\infty dx_1 \int_{-\infty}^\infty dx_2 \, [\, W(x_1, x_2) \\ \qquad - \int_{-\infty}^\infty dy_1 \int_{-\infty}^\infty dy_2 \, P(x_1, x_2 \mid y_1, y_2) \, W(y_1, y_2)] \end{array} \right)}{\int_0^\infty dx_1 \int_{-\infty}^\infty dx_2 \, W(x_1, x_2)} \tag{76}$$

The basic ideas to find an approximate solution of the master equation (74) are the same as for the white-noise case in Section 3. However, the detailed arguments and calculations are considerably more involved and we only give here some main results, valid for sufficiently small noise strengths $\sigma_i$ and parameters $\Delta$ under the additional condition (41) with $(\sigma_1^2 + \sigma_2^2)^{1/2}$ instead of $\sigma$: For $\varepsilon \leqslant x \leqslant 1 - \varepsilon$ one finds analogously to (47) that

$$W(x, y) = \rho(x) \, \tilde{\rho}(y) \tag{77}$$

where $\rho(x)$ and $\tilde{\rho}(y)$ are defined in (43) and (72), respectively. For $x \geqslant 1 - \varepsilon$ Eq. 23 goes over into

$$W(x, y) = \frac{2\rho(1/2)}{\sqrt{\pi z b^{1/z}}} \tilde{\rho}(y) [2(\sigma_1^2 + \sigma_2^2)]^{(1-z)/2z} \, H\left( \frac{1 + \Delta - x + Ay}{[2(\sigma_1^2 + \sigma_2^2)]^{1/2}} \right) \tag{78}$$

For $x \leqslant \varepsilon$ the quasi-invariant density $W(x, y)$ can be written as an infinite sum similarly to (33) with summands $w_i(x, y)$ which are given by rather complicated expressions for general $i$. Only in the limit of large $i$ does one again arrive at a simple result similar to (36):

$$w_i(x, y) = \frac{2\rho(1/2)}{\sqrt{\pi z b^{1/z} u^i}} \tilde{\rho}(y) \, S^{(1-z)/z} H\left( \frac{x + \Delta u^i}{u^i S} \right) \tag{79}$$

$$S := \left[ 2 \frac{u^2}{u^2 - 1} \left( \sigma_1^2 + \frac{\sigma_2^2}{1 - A^2} \frac{u - 3A + 4A/u}{u - A} \right) \right]^{1/2} \tag{80}$$

It can be readily seen that $u - 3A + 4A/u > 0$ for all $u > 1$ and $-1 < A < 1$. Note that $\sigma_1^2$ and $\sigma_2^2/(1 - A^2)$ in (80) are equal to the variances $\langle (\xi_n^{(1)})^2 \rangle$ and $\langle y_n^2 \rangle$ of the noises in the Langevin equation (68) according to (70) and (73), respectively.

It is remarkable that both in (77) and (79) we find a factorization into functions of one variable only. However, the same is not the case in (78) nor for the functions $w_i(x, y)$ with small $i$ and thus for the sum $W(x, y)$ unless $A = 0$ or $\sigma_2 = 0$. It should also be noted that for nonvanishing $A$ and

$\sigma_2$ Eqs. (77)–(79) are valid only for $y$ values close to $y=0$, say $-\tilde{\varepsilon} \leqslant y \leqslant \tilde{\varepsilon}$, where $\tilde{\varepsilon}$ is a small quantity similar to $\varepsilon$. Although explicit results cannot be given for $|y| > \tilde{\varepsilon}$, in the domain $\varepsilon \leqslant x \leqslant 1-\varepsilon$ a factorization as in (77) will certainly not occur unless $A=0$ or $\sigma_2=0$. Furthermore, one can show that for sufficiently small $\sigma_i$ and $\Delta$ the contributions to integrals over $y$ as for example in the rate (76) from $y$ values outside $[-\tilde{\varepsilon}, \tilde{\varepsilon}]$ are so small that using (77)–(79) on the whole $y$ axis still leads to arbitrarily good approximations.

By construction, the functions $w_i(x, y)$ fulfill a two-dimensional version of (38). Then, the rate (76) can be evaluated by the same line of reasoning as in Section 4, leading to the generalized versions of (52) and (58):

$$k = \rho\left(\frac{1}{2}\right)\left(\frac{S}{b}\right)^{1/z} F\left(\frac{\Delta}{S}\right) = \rho\left(\frac{1}{2}\right)\left(\frac{\Delta}{b}\right)^{1/z} G\left(\frac{S}{\Delta}\right) \tag{81}$$

where $S$ is defined in (80). The discussion of this result is very similar to that in Section 4, and in Fig. 6 it is shown that it compares excellently with numerical simulations. From the definition of $S$ in (80) and the properties of $G(x)$ in (59) it follows that the rate (81), considered as a function of $A$, may be increasing, decreasing, or exhibit one or several extrema, depending on the values of $\Delta$, $\sigma_i$, and $u$. The latter case of a nonmonotonous dependence of the rate on the correlation time of the colored noise has recently attracted much attention under the name 'resonant activation';[38] see also ref. 37 and further references therein.

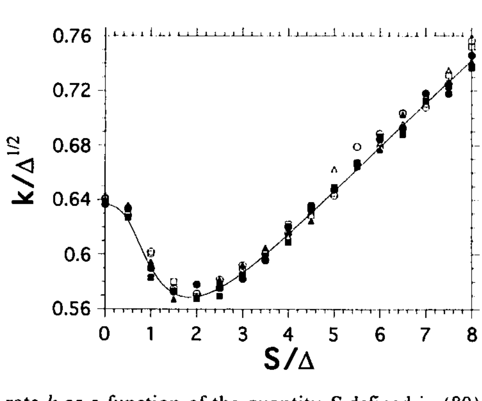


Fig. 6. The escape rate $k$ as a function of the quantity $S$ defined in (80) for the logistic map (8) in the presence of colored noise (68). The solid line is the theoretical prediction according to (9), (45), and (81). The symbols are results from numerical simulations of the Langevin equations (68), (69) with $\sigma_1 = 0$. The parameter values are $\Delta = 10^{-6}$ (open symbols), $\Delta = 10^{-7}$ (filled symbols), and $A=0$ (circles), $A=0.5$ (squares), and $A=-0.5$ (triangles).

## 7. SUMMARY AND DISCUSSION

We investigated one-dimensional maps near the boundary crisis at fully developed chaos in the presence of weak Gaussian noise. We introduced a new method to calculate the quasi-invariant density $W(x)$ which becomes asymptotically exact for small noise strengths $\sigma$ and distances $\Delta$ from the crisis. For simplicity, in the detailed discussion of this method in Section 3 we restricted ourselves to additive white noise and single-humped maps which are symmetric about the maximum of order $z>0$ at $x=1/2$ and linear below the unstable fixed point at $x=0$; see Fig. 1. The extension to more general maps and noises, including multiplicative and colored noise, was briefly outlined in Sections 5 and 6, while non-Gaussian distributed noise will be treated in a subsequent publication.[22]

Our method for the determination of the quasi-invariant density $W(x)$ can be considered as a kind of perturbation theory in *two* small parameters $\sigma$ and $\Delta$. The unperturbed invariant density $\rho(x)$ at the crisis $\Delta=0$ in the absence of noise $\sigma=0$ is thus assumed to be known. In fact, $W(x)$ coincides with $\rho(x)$ for all $\varepsilon \leqslant x \leqslant 1-\varepsilon$, where $|\Delta|, \sigma \ll \varepsilon \ll 1$; see Eq. (47). It is only close to the boundaries of the unit interval where $\rho(x)$ has singularities, that $W(x)$ behaves substantially differently, staying smooth and finite, and outside the unit interval, where $\rho(x)$ vanishes, while $W(x)$ stays small but finite. More precisely, $W(x)$ is given by Eq. (23) for $x \geqslant 1-\varepsilon$ and by Eqs. (33), (36) for $x \leqslant \varepsilon$; see also Fig. 3. Thus small variations of $\Delta$ or $\sigma$ may lead not to small perturbations, but to substantial qualitative changes of $W(x)$ in certain $x$ domains. In particular, our perturbation-like approach does not fit into the framework of a linear response theory.[26, 39] Rather, it reminds one of singular perturbation theory for differential equations[40] so far as the properties of the solutions are concerned, while the approach itself is completely different.

Besides the assumption of small parameters $\Delta$ and noise strengths $\sigma$, we derived an additional necessary and sufficient condition (41) for the validity of the above results for $W(x)$. Under this condition, our method overcomes the difficulties of any perturbation theory about fully developed chaos as discussed in detail by Grossmann,[26] particularly with respect to a linear response theory for a deterministic map dynamics.[39]

Our investigation is complementary to the work of Bene and Szépfalusy.[7] They concentrated on weak multiplicative noises and small perturbations of the map at the crisis, which only lead to small changes of the invariant density for all $x$ values in comparison with the unperturbed system in the absence of noise and thus allow a linear response theory.[26] The approach presented here is also complementary to the path-integral and WKB-like methods for maps with weak noise put forward in refs. 8, 24, 29, 31, 32, and 41: These methods are a powerful tool provided the noise strength $\sigma$ is sufficiently small in comparison with any other parameter of the system. In particular, for the problem considered here, they only apply for $\Delta$ values with $\Delta \ll -\sigma$. On the other hand, the condition (41) must no longer be fulfilled. It can be shown that for small $\sigma$ and $\Delta$ fulfilling both $\Delta \ll -\sigma$ and (41) these methods lead to the same results as the approach presented here. Finally, it should be mentioned that concepts somewhat related to those in Section 3 have also been developed in ref. 42. However, the context in which these concepts are used is rather different from ours.

From the quasi-invariant density we derived the central rate formula (52), which has the form of a scaling law and compares excellently with the results from numerical simulations shown in Tables I and II and in Fig. 5. Generalizations of this rate formula were given Sections 5 and 6, in particular for an extended class of single-humped maps $f(x)$ with multiplicative noise (60) in (61) and with colored noise (68) in (81); see also Fig. 6. Under certain conditions, a nonmonotonic dependence of the rate on the correlation time of the colored noise may occur, a phenomenon which has recently attracted considerable attention under the label 'resonant activation'.[38] Moreover, within this extended class of single-humped maps and Gaussian noises we found that in the scaling law for the rate both the critical exponents and the scaling functions are universal for any fixed order $z > 0$ of the maximum of the maps. In other words, for fixed $z$ and Gaussian distributed noise, any further details of the map and the noise enter into the scaling law of the rate only through the scaling amplitudes.[33]

By considering the escape rate for fixed parameter $\Delta > 0$ as a function of the noise strength $\sigma$ we found that deterministic transient chaos is stabilized by sufficiently weak but finite noise for any single-humped map with a maximum of order $z > 1$; see Fig. 4. As detailed in refs. 20 and 22, there is no simple intuitive explanation of this effect except in the limit $f'(0) \to \infty$ corresponding to a superunstable fixed point or a discontinuity of the map $f(x)$ at $x = 0$. For fixed $z > 1$, the strongest possible noise-induced reduction $\min_\sigma k(\sigma)/k(\sigma = 0)$ of the deterministic escape rate $k(\sigma = 0)$ is universal, in particular independent of the parameter $\Delta$, and monotonically decreases as a function of $z$ from 1 for $z = 1$ to 1/2 for $z \to \infty$. Numerical evidence and a heuristic argument for noise-induced stabilization of deterministic transient chaos for noisy maps in more than one dimension near a particular boundary crisis called unstable–unstable pair bifurcation was also given in ref 5. While the effect seems to be more pronounced in the latter case, it is of broader universality in our case.

Generalizations of the methods introduced here to maps near interior crises[1] and band-merging points and to systems in more than one dimension are under investigation.

## APPENDIX A. SUFFICIENT CONDITION FOR (42)

In this appendix we show that (42) holds in arbitrarily good approximation for sufficiently small $\sigma$ and $\Delta$ provided the condition (41) is fulfilled. We always assume that $x \in [\varepsilon/u, \varepsilon]$. Thus, in particular, $x > 0$. We first show that if it is possible to find an integer $N$ such that

$$N \gg 1 \qquad \text{and} \qquad \varepsilon \gg u^N(\sigma U_\infty + |\Delta|) \tag{A1}$$

is fulfilled sufficiently well, then one has in arbitrarily good approximation

$$\sum_{i=1}^{N-1} w_i(x) = \frac{B}{u^{1/z} - 1} x^{(1-z)/z} \tag{A2}$$

Since such an $N$ can always be found for sufficiently small $\sigma$ and $\Delta$, we will be left to find conditions such that the contribution of $\sum_{i=N}^{\infty} w_i(x)$ in (33) is negligible in comparison with (A2) in order to arrive at (42). To prove (A2), we first note that (A1) implies $x \gg u^i|\Delta|$ and $x + u^i\Delta \gg \sigma u^i U_i$ for all $i < N$ [according to (37) we have $\sqrt{2} = U_0 \leqslant U_i \leqslant U_\infty < \infty$]. Thus (39) applies and the term $u^i\Delta$ on the right-hand side can be neglected. With the approximation $1 - u^{-N/z} = 1$ this yields (A2).

Next we address the sum $\sum_{i=1}^{\infty} w_i(x)$ in the case $z > 1$. We consider separately the following two possibilities: (i) If $\Delta \gg \sigma U_\infty$, then we have $x + u^i\Delta \gg u^i U_\infty \sigma$ for all $i \geqslant N$. Thus again (39) applies and it immediately follows that $\sum_{i=N}^{\infty} w_i(x)$ is negligible in comparison with (A2) for sufficiently large $N$. (ii) If $\Delta$ is not very much larger than $\sigma U_\infty$ or negative, say $\Delta \leqslant D\sigma U_\infty$ with $D \gg 1$ fixed, then we estimate each $w_i(x)$ from above by its maximum, which is approximately given by the right-hand side of (40). This implies that

$$0 \leqslant \sum_{i=N}^{\infty} w_i(x) \leqslant \sum_{i=N}^{\infty} \frac{C}{u^i(\sigma U_i)^{(z-1)/z}} \leqslant \frac{C}{1 - u^{-1}} \frac{1}{u^N(\sigma U_0)^{(z-1)/z}} \tag{A3}$$

It follows that the sum $\sum_{i=N}^{\infty} w_i(x)$ is certainly negligible in comparison with (A2) if

$$u^N \sigma^{(z-1)/z} \gg \varepsilon^{(z-1)/z} \tag{A4}$$

It is straightforward to show that (A1) and (A4) can always be fulfilled simultaneously for sufficiently small $\sigma$ and $\Delta$ if $0 \leqslant \Delta \leqslant D\sigma U_\infty$. If $\Delta < 0$, the same follows under the (sufficient) extra condition (41).

In the case $z \leqslant 1$ we note that for any fixed $x$ and $i$ the function $w_i(x)$ in (36) is monotonically increasing with $\Delta$. It thus is sufficient to estimate the sum $\sum_{i=N}^{\infty} w_i(x)$ for $\Delta \gg \sigma U_\infty$. As before, this implies that (39) can be used, yielding

$$\sum_{i=N}^{\infty} w_i(x) = B \sum_{i=N}^{\infty} u^{-i/z}[x + u^i \Delta]^{(1-z)/z} \tag{A5}$$

It is straightforward to show that for $x \in [\varepsilon/u, \varepsilon]$the sum (A5) is negligible in comparison with (A2) if (A1) is fulfilled.

## APPENDIX B. NECESSARY CONDITION FOR (42)

In this appendix we construct maps $f(x)$ with arbitrary $z > 1$ within periodic windows arbitrarily close to $\Delta = 0$ such that (42) is violated for $\Delta \ll -\sigma^{(z-1)/z}$. It has been shown in ref. 43 for the case $z = 2$ and can be readily generalized to any $z > 1$ that there exist families of maps parametrized by $\Delta$ as introduced at the beginning of Section 2 with the following properties: one can find negative parameter values $\Delta$ inside certain periodic windows arbitrarily close to zero such that the periodic attractor $\{x_1^* = f(x_p^*), x_2^* = f(x_1^*), ..., x_p^* = f(x_{p-1}^*)\}$ of $f(x)$ has one element, say $x_1^*$, very close to the maximum $x = 1/2$ of $f(x)$ and obeys $x_{i+1}^* > x_i^*$ for $3 \leqslant i \leqslant p$, where $x_{p+1}^* := x_1^*$, see Fig. 7a. Since $x_1^*$ is very close to $x = 1/2$, it follows with (3) that $x_2^* = f(x_1^*)$ is very close to $x = 1$ and thus the linearization (5) applies for all $x \geqslant x_2^*$. Similarly, due to (4) and (5) the points $x_3^*, x_4^*, ..., x_l^*$ up to a certain $l \leqslant p$ are close to $x = 0$ and

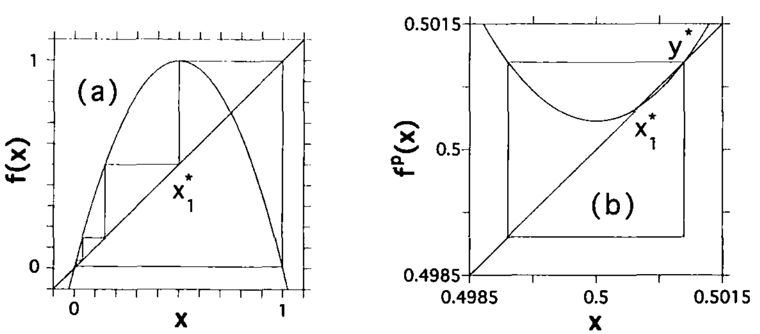


Fig. 7. Example of a single-humped map $f(x)$ inside a periodic window as considered in Appendix B. (a) The logistic map (8) at $\Delta = -0.0024354026$, the bisectrix $y = x$, and the stable periodic orbit $\{x_1^*, ..., x_5^*\}$ of period $p = 5$. (b) A detail of the iterate $f^p(x)$ close to $x = 1/2$, the bisectrix $y = x$, and the element $x_1^*$ of the stable periodic orbit. The points $x_1^*$ and $y^* = 1/2 + \eta$ are the unique stable and unstable fixed points of $f^p(x)$ close to $x = 1/2$. The interval $[1/2 - \eta, 1/2 + \eta]$ is mapped into itself by $f^p(x)$, as indicated by the square. The parameter $\Delta$ has been chosen such that the stable fixed point obeys $x_1^* = 1/2 + \eta/2$; see (B6).

thus the linearization (4) applies for all $x \leqslant x_i^*$. In the following we restrict ourselves to maps $f(x)$ for which the linearization (4) is a reasonable approximation even for all $x \leqslant x_p^*$, such as in Fig. 7a. Since $x_1^* = f(x_p^*)$ is very close to $x = 1/2$, this implies that $x_p^*$ is close to $x = 1/(2u)$. In other words, the linearization (4) is assumed to be a reasonable approximation for all $x \leqslant 1/(2u)$ independent of $p$. It is obvious that even within this additional assumption regarding $f(x)$ we are still dealing with a generic situation.

For $x$ values close to $x = 1/2$ and thus to $x_1^*$ it follows that $f^i(x)$ [denoting the $i$ th iterate of $f(x)$] is close to $f^i(x_1^*) = x_{i+1}^*$ for $1 \leqslant i \leqslant p-1$. Thus, $f^2(x) = u(1 - f(x))$ according to (5). Similarly, one has $f^i(x) = uf^{i-1}(x)$ for $3 \leqslant i \leqslant p$ according to (4). With (3) we then find for $x$ close to $x = 1/2$ that

$$f^p(x) = f^{p-2}(u[1 - (1 + \Delta - b\,|x - 1/2|^z)]) = -u^{p-1}(\Delta - b\,|x - 1/2|^z) \quad \text{(B1)}$$

Thus, near $x = 1/2$ the iterate $f^p(x)$ is a single-humped map with a local minimum of order $z$ at $x = 1/2$ and is symmetric about this point; see Fig. 7b. The point $x_1^*$ is a stable fixed point of $f^p(x)$. By differentiation of (B1) this yields

$$-1 < \frac{d}{dx} f^p(x_1^*) = \prod_{i=1}^{p} f'(x_i^*) = u^{p-1} \frac{zb\,|x_1^* - 1/2|^z}{x_1^* - 1/2} < 1 \quad \text{(B2)}$$

Since $x_1^* = f^p(x_1^*)$ is very close to $x = 1/2$, we have in very good approximation that $1/2 = f^p(1/2)$ and thus we can conclude from (B1) and (B2) that

$$\begin{aligned} \frac{1}{2} &= -u^{p-1}\left(\Delta - b\left|x_1^* - \frac{1}{2}\right|^z\right) \\ &= -u^{p-1}\Delta + u^{-z(p-1)}\left|\frac{1}{zb}\frac{d}{dx} f^p(x_1^*)\right|^{z/(z-1)} \end{aligned} \quad \text{(B3)}$$

By means of the inequalities in (B2) it follows from (B3) for sufficiently large $p$ that

$$\tfrac{1}{2} = -u^{p-1}\Delta \quad \text{(B4)}$$

Thus, small negative parameters $\Delta$ are equivalent to large periods, which is quite obvious of course.

Since $x_1^*$ is the only stable fixed point of $f^p(x)$ near $x = 1/2$, the intersection of $f^p(x)$ in (B1), with the bisectrix $y = x$ closest to $x = 1/2$ is at $x = x_1^*$ with a slope of modulus smaller than one; see (B2). Moreover, (B1) implies that close to $x=1/2$ there must be exactly one further intersection point

$$y^* = 1/2 + \eta, \qquad 0 < \eta \ll 1 \tag{B5}$$

of $f^p(x)$ and the bisectrix $y = x$. Clearly, $y^*$ is an unstable fixed point of $f^p(x)$ with $(d/dx)f^p(x=y^*) > 1$ and the interval $[1/2-\eta, 1/2+\eta]$ is mapped into itself by $f^p(x)$; see Fig. 7b. Inside a particular periodic window we are still free to choose a value of $x_1^* - 1/2$ between $\eta$ (where the periodic window emerges by tangent bifurcation) and a certain $\zeta$, $-\eta < \zeta < 0$ (where the first period-doubling bifurcation inside the periodic window occurs). For later convenience, we make the following choice:

$$x_1^* = 1/2 + \eta/2 \tag{B6}$$

See Fig. 7b. Then, it follows from (B2) that $0 < (d/dx) f^p(x_1^*) < 1$. Furthermore, closer inspection of (B1) shows that by fixing the relative distances of the fixed points $x_1^*$ and $y^*$ from $x = 1/2$ through (B5) and (B6), the derivative $(d/dx) f^p(x)$ at $x = x_1^*$ becomes independent of $p$ and hence of $\Delta$. Similar universality properties of periodic windows have been studied in ref. 44.

Next we use some results from the theory of generalized potentials[31, 32] or quasipotentials[24] for maps disturbed by weak noise. According to this theory, the quasi-invariant density can be written as

$$W(x) = Z_\sigma(x)\, e^{-\phi(x)/2\sigma^2} \tag{B7}$$

where the so-called generalized potential $\phi(x)$ is independent of the noise strength $\sigma$ and the $\sigma$ dependence of the prefactor $Z_\sigma(x)$ is weaker than exponential. In other words, for sufficiently small noise strengths the 'Boltzmann factor' $e^{-\phi(x)/2\sigma^2}$ dominates the behavior of the quasi-invariant density $W(x)$ in (B7). In the crease of a periodic window as we consider it here, the generalized potential has the following properties: First, it has a local quadratic minimum at $x_1^*$ of the form[32]

$$\phi(x_1^* + \delta x) = \frac{\phi''(x^*)}{2}\, \delta x^2 + O(\delta x^3) \tag{B8}$$

where $\phi''(x_1^*)$ is given by

$$\phi''(x_1^*) = \frac{2\left(1 - \prod_{i=1}^{p} f'(x_i^*)^2\right)}{1 + f'(x_p^*)^2 + f'(x_p^*)^2 f'(x_{p-1}^*)^2 + \cdots + \prod_{i=2}^{p} f'(x_i^*)^2} \tag{B9}$$

Using the approximations (4) for $x \leqslant x_p^*$ and (5) for $x \geqslant x_2^*$, we can infer that

$$\phi''(x_1^*) = \frac{2(1 - \prod_{i=1}^{p} f'(x_i^*)^2)(u^2 - 1)}{u^{2p} - 1} \tag{B10}$$

Second, for our choice (B6) of $x_1^*$, neglecting the term $O(\delta x^3)$ in (B8) yields a reasonably good approximation for $\phi(x)$ on the whole interval $[x^*, y^*]$.[32] Third, the generalized potential is constant for $x$ values between $y^*$ and a small neighborhood of $x_2^*$ analogous to the neighborhood $[1/2 - \eta, 1/2 + \eta]$ of $x_1^*$.[24] The behavior of $\phi(x)$ for $x < x_1^*$ is similar but not further needed here. Thus, we find a potential difference between $x = x_1^*$ and the plateau beyond $x = y^*$ which is approximately given by

$$\Delta\phi = \phi(y^*) - \phi(x_1^*) = \frac{\phi''(x_1^*)}{2}(y^* - x_1^*)^2 = \frac{\phi''(x_1^*)}{2}\left(x_1^* - \frac{1}{2}\right)^2 \tag{B11}$$

where the last equality follows from (B5) and (B6). By means of (B2), (B4), and (B10) we can infer that

$$\Delta\phi = \frac{(1 - \prod_{i=1}^{p} f'(x_i^*)^2)(u^2 - 1)}{(u/2\Delta)^2 - 1}\left|\frac{2\Delta}{zb}\prod_{i=1}^{p} f'(x_i^*)\right|^{2/(z-1)} \tag{B12}$$

As mentioned below (B6), the product $\prod_{i=1}^{p} f'(x_i^*) = (d/dx) f^p(x_1^*)$ is a $p$- and thus $\Delta$-independent quantity larger than zero and smaller than one. Hence, for small $\Delta$ we have $\Delta\phi = const \cdot |\Delta|^{2z/(z-1)}$, where $const$ is a $\Delta$-independent positive constant. From (B7) and (B11) it then follows that

$$\frac{W(x_1^*)}{W(y^*)} = \frac{Z_\sigma(x_1^*)}{Z_\sigma(y^*)} \exp\left\{-\frac{\mathbf{const}}{2}\left(\frac{|\Delta|^{z/(z-1)}}{\sigma}\right)^2\right\} \tag{B13}$$

Since $Z_\sigma(x)$ is nonexponential in $\sigma$, the quasi-invariant density $W(x)$ becomes strongly $\sigma$ dependent near $x = 1/2$ for $\sigma \ll |\Delta|^{z/(z-1)}$. On the other hand, in Section 3.3 it is shown that under the assumption (42) one finds a $\sigma$-independent $W(x)$ near $x = 1/2$; see Eq. (47). Thus, (42) cannot be valid for $\Delta \ll -\sigma^{(z-1)/z}$.

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